PMM U.S.S.R., Vol.53, No.3, pp.368-374, 1989
Printed in Great Britain
0021-8928/89 $10.00+0.00

ANALYSIS OF CONTROLLED ROTATIONS OF AN ELASTIC ROD AROUND AN ARBITRARY AXIS*

L.D. AKULENKO and S.A. MIKHAILOV

Controlled plane rotations around an arbitrary axis under the action of concentrated (transverse) forces and the moments of forces /2, 3/ are investigated for a homogeneous rectilinear rod within the framework of the linear theory of small strains /1/. To be specific, we consider the case when the moments are applied relative to the ends of the rod and the axis of rotation while forces orthogonal to the rod are also concentrated at the ends. A complete solution of the Cauchy problem (in time) is constructed by analytic methods of mathematical physics /4/ and the approach in /5/ for a system described by the boundary value problem, and a foundation is given /6, 7/ and controllability is established /8/. Problems of applied interest are examined in the cases of realizations of the "kinematic" and "dynamic" controls of rod rotations and estimates of the errors due to its elasticity are also examined. The results can be utilized in investigations of problems of precision control of mechanical objects possessing substantial compliance of the structure.

1. ***Mechanical model and formulation of the problem.*** Plane rotations of an inextensible elastic rod around an axis OZ fixed in inertial $OXYZ$ space and passing through an arbitrary fixed point O of a rod AB are considered (Fig.1). It is assumed that the rod is recilinear in the undeformed state while its characteristics, the length l, the linear density ρ, and the bending stiffness EI, where E is the Young's modulus of the material and I is the moment of inertia of the transverse section, are constants. The rod AB is subjected to the action of external concentrated forces P_A, P_B (at the moving points A, B) and moments of the forces M_0, M_A, M_B (relative to the fixed axis OZ and the moving axes AZ, BZ orthogonal to the OXY plane) (Fig.1). The elastic displacements are assumed to be small, allowing investigation in a linear approximation /1/. Under these conditions only normal force components to the rod P_A and P_B should be taken into account.

On the basis of the assumptions mentioned, that are often used in practice, the equations of absolute motion of an elastic rod AB in the OXY plane can be represented in the form /1/

$$\rho u^{\cdot\cdot} = -EIu^{\mathrm{IV}},\ u = u(t, x) \tag{1.1}$$

$$u(t, x) = \begin{cases} u_a(t, x), & x \in [-a, 0] \\ u_b(t, x), & x \in [0, b] \end{cases}$$

$$t \in [0, T],\ T < \infty;\ x \in [-a, b],\ a + b = l$$

The desired unknown function $u(t, x)$ governing the total displacement of an arbitrary point $G \in AB$ whose coordinate is x, can be interpreted as the sum of two quantities

$$u(t, x) = \varphi(t)x + v(t, x) \tag{1.2}$$

Here $\varphi = \varphi(t)$ is the unknown angle between the fixed OX axis and the rotating coupled axis Ox, i.e., φx is the length of the appropriate arc (Fig.1). It is assumed for convenience in the investigation that the Ox axis coincides with the tangent to the rod AB at the point $O \in AB$ $(x = 0)$. The unknown variable $v(t, x)$ has the meaning of small relative elastic displacements of the point $G \in AB$ of the rod with Euler coordinate x, $x \in [-a, b]$.

The boundary conditions at the points A, O, B (for $x = -a$, 0, b, respectively) have the following natural form

$$u_a(t, 0) = u_b(t, 0) = 0, \quad u_a'(t, 0) = u_b'(t, 0) \tag{1.3}$$

$$-EIu_a''(t, -a) = M_A(t), \quad -EIu_b''(t, b) = M_B(t)$$

*Prikl. Matem. Mekhan., 53, 3, 476-484, 1989

$$-EIu_a'''(t, -a) = P_A(t), \quad -EIu_b'''(t, b) = P_B(t)$$
$$-EI[u_b''(t, 0) - u_a''(t, 0)] = M_0(t) \tag{1.4}$$

The first three conditions in (1.3) have a geometrical meaning while the remaining five conditions of (1.4) (of eight relationships) describe the influence of the concentrated external forces and moments of forces as mentioned above.

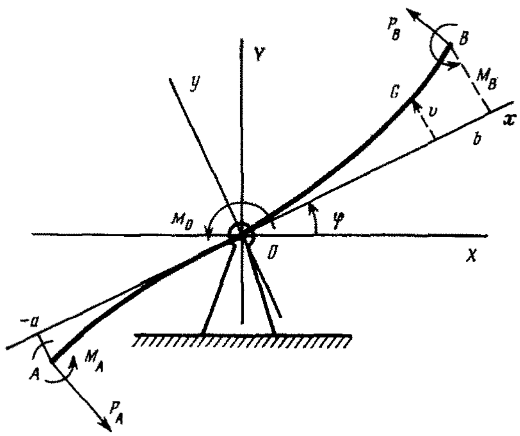

Fig.1

The external actions $M_{O,A,B}$, $P_{A,B}$ are considered to be given fairly smooth functions of the time, i.e., to belong to a certain class of allowable functions determined later; their realization is by ideal modes. We note that if the solution $u(t, x)$ of the boundary value problem (1.1), (1.3) and (1.4) corresponding to the allowable initial conditions is found, then according to (1.2) the variables $\varphi(t)$ and $v(t, x)$ are determined uniquely in the following manner /2/:

$$\varphi(t) = u'(t, 0), \quad v(t, x) = u(t, x) - u'(t, 0)x \tag{1.5}$$

The initial conditions have a general form and according to (1.3) satisfy the conjugate conditions

$$u(0, x) = f(x), \quad u^{\cdot}(0, x) = g(x), \quad x \in [-a, b]$$
$$f(x) = f_a(x), \quad g(x) = g_a(x), \quad x \in [-a, 0]$$
$$f(x) = f_b(x), \quad g(x) = g_b(x), \quad x \in [0, b] \tag{1.6}$$
$$f(0) = f_a(0) = f_b(0) = 0, \quad f_a'(0) = f_b'(0)$$
$$g(0) = g_a(0) = g_b(0) = 0, \quad g_a'(0) = g_b'(0)$$

Here $f(x) = f_{a,b}(x)$, $g(x) = g_{a,b}(x)$ are known fairly smooth functions that should be almost linear according to (1.5).

The boundary value problem (1.1), (1.3) and (1.4) must be solved under the given allowable external actions (1.4) and initial distributions (1.6) for arbitrary values of the parameter a ($a \in [0, l]$, $a + b = l$).

Without loss of generality it is later possible to set $l = \rho = EI = 1$, which is achieved by inserting dimensionless arguments and variables according to the formulas

$$t^* = \nu t, \quad \nu^2 = EI/(\rho l^4), \quad x^* = x/l, \quad u^* = u/l \tag{1.7}$$
$$M^*_{O,A,B} = M_{O,A,B}/(\rho l^3\nu^2), \quad P^*_{A,B} = P_{A,B}/(\rho l^2\nu^2)$$
$$f^*(x^*) = f(lx^*)/l, \quad g^*(x^*) = g(lx^*)/(l\nu), \quad a^* = a/l, \quad b^* = b/l$$

For brevity, the asterisk is henceforth omitted.

2. Solution of the eigenvalue and eigenfunction boundary value problem.

The Fourier method /4/ and Grinberg's procedure /5/ are used to construct the desired solution $u(t, x)$ by analogy with /2, 3/. First, systems of eigenvalues and eigenfunctions are constructed by separation of variables for the appropriate spatial variable x of the self-adjoint boundary value problem

$$S^{IV} - \lambda^4 S = 0, \quad S = S(x) = S_{a,b}(x), \qquad x \in [-a, b] \tag{2.1}$$
$$S_{a,b}(0) = 0, \quad S_a'(0) = S_b'(0); \quad S_a''(0) = S_b''(0)$$
$$S_a''(-a) = S_a'''(-a) = S_b''(b) = S_b'''(b) = 0$$
$$(u(t, x) \sim \Theta(t) S(x), \quad \Theta^{\cdot\cdot} + \lambda^4\Theta = 0, \quad t \in [0, T])$$

We can simplify the finding and writing of the system of real eigenvalues $\{\lambda\}$ and eigenfunctions $\{S_{a,b}(x)\}$ if new spatial arguments x_a, x_b are introduced in the appropriate ranges of their variation as follows $x_a = a + x$, $x_a \in [0, a]$; $x_b = b - x$, $x_b \in [0, b]$; then $S_a(x) = S_a(x_a - a) \equiv Q_a(x_a)$, $S_b(x) = S_b(b - x_b) \equiv Q_b(x_b)$. According to (2.1)

$$Q_{a,b}(x_{a,b}) = A_{a,b}\cos\lambda x_{a,b} + B_{a,b}\sin\lambda x_{a,b} + \tag{2.2}$$
$$C_{a,b}\operatorname{ch}\lambda x_{a,b} + D_{a,b}\operatorname{sh}\lambda x_{a,b}; \quad A_{a,b} = C_{a,b}, \quad B_{a,b} = D_{a,b}$$

The four relationships (2.2) between the constant coefficients $A_{a,b}$ and $C_{a,b}$, $B_{a,b}$ and $D_{a,b}$ follow from the zero conditions (2.1) at the ends of the rod, i.e., for $x_{a,b} = 0$. The four conditions of (2.1) for $x = 0$, i.e., $x_a = a, x_b = b$, are utilized to determine $A_{a,b}$ and $D_{a,b}$. It is convenient to reduce these conditions to the following relationships between D_a and D_b by elementary manipulations:

$$D_a[q(\alpha) - s(\alpha)r(\alpha)/q(\alpha)] = -D_b[q(\beta) - s(\beta)r(\beta)/q(\beta)]$$
$$D_a[r(\alpha) - p(\alpha)s(\alpha)/q(\alpha)] = D_b[r(\beta) - p(\beta)s(\beta)/q(\beta)]$$
$$A_a = -D_a s(\alpha)/q(\alpha), \quad A_b = -D_b s(\beta)/q(\beta) \tag{2.3}$$
$$p(\xi) = \operatorname{ch}\xi - \cos\xi, \quad q(\xi) = \operatorname{ch}\xi + \cos\xi \geqslant 2$$
$$r(\xi) = \operatorname{sh}\xi - \sin\xi, \quad s(\xi) = \operatorname{sh}\xi + \sin\xi$$
$$\xi = \alpha \vee \beta; \quad \alpha = \lambda a, \quad \beta = \lambda b$$

We obtain a characteristic equation determining the eigenvalues $\lambda = \lambda(a, b)$: from the first two equations in D_a and D_b

$$\Delta(\alpha, \beta) = [q^2(\alpha) - s(\alpha)r(\alpha)][r(\beta)q(\beta) - p(\beta)s(\beta)] +$$
$$[q^2(\beta) - s(\beta)r(\beta)][r(\alpha)q(\alpha) - p(\alpha)s(\alpha)] = 0$$
$$\{\lambda\} = \{\lambda_n(a, b)\}, \; n = 0, \pm 1, \pm 2, \ldots, \lambda_0 \equiv 0 \tag{2.4}$$

The function $\Delta(\alpha, \beta)$ in (2.4) is an odd function of λ relative to $\lambda = 0$; consequently $\lambda_0 = 0$ and it can later be considered that $\lambda_{-n} = -\lambda_n$ $(n = 0, 1, 2, \ldots)$, where λ_n are non-negative roots of the characteristic equation. The zeroth root $\lambda_0 = 0$ is multiple, however, only the first eigenfunction $S_0(x) = x$ corresponds to it. The non-zero eigenvalues λ_n are simple and symmetric relative to commutation of the arguments a and b $\lambda_n(a, b) = \lambda_n(b, a)$ $(n \geqslant 1, a + b = 1)$. Consequently, it is sufficient to construct functions $\lambda_n(a, b)$, solutions of (2.4) in the interval $a \in [0, {}^1/_2]$, $b = 1 - a$ (or $b \in [0, {}^1/_2]$, $a = 1 - b$).

It should be noted that the eigenvalues λ_n^f agree with the corresponding values of the boundary value problem for a rod, hinge supported at one end /2, 3/, for $a = 0$ $(b = 1)$ or $a = 1$ $(b = 0)$ $(\lambda_n(0, 1) = \lambda_n(1, 0))$ as is obvious.

Thus, we have the relationships

$$\operatorname{tg}\lambda = \operatorname{th}\lambda \qquad (a = 0, \; b = 1 \vee a = 1, \; b = 0) \tag{2.5}$$
$$\lambda_1^f \simeq 3.927, \; \lambda_2^f \simeq 7.069, \quad \lambda_3^f \simeq 10.21, \; \lambda_4^f \simeq 13.35, \quad \lambda_5^f \simeq 16.49$$
$$\lambda_n^f = \pi/4 + \pi n + O_n^f, \quad O_n^f \sim \exp[-(2\pi n + \pi/2)], \quad n \gg 1$$

The case $a = b = {}^1/_2$ for which we obtain the characteristic equation and its solution

$$(\operatorname{tg}{}^1/_2\lambda - \operatorname{th}{}^1/_2\lambda)(\cos{}^1/_2\lambda\operatorname{ch}{}^1/_2\lambda + 1) = 0 \qquad (a = b = {}^1/_2) \tag{2.6}$$
$$\{\lambda_n\} = \{\lambda_{n'}^s\} \cup \{\lambda_{n''}^a\}, \quad n' = 0, 1, 2, \ldots, \quad n'' = 1, 2, \ldots$$
$$\lambda_1^a \simeq 3.750, \quad \lambda_2^a \simeq 9.388, \quad \lambda_3^a \simeq 15.71, \quad \lambda_4^a \simeq 21.99, \quad \lambda_5^a \simeq 28.27$$
$$\lambda_{n''}^a = \pi + 2\pi n'' + O_{n''}^a, \quad O_{n''}^a \sim \exp(-2\pi n''), \quad n'' \gg 1$$

is also investigated fairly simply.

The eigenvalues $\{\lambda_{n'}^{s}\}$ corresponding to symmetric rod vibrations modes (zeros of the first factor in (2.6)) are obtained on the basis of (2.5): $\lambda_{n'}^{s} = 2\lambda_{n'}(0, 1)$. These eigenvalues and the eigenvalues $\{\lambda_{n''}^{a}\}$, corresponding to the antisymmetric rod vibration modes (zeros of the second factor in the left side of (2.6)) alternate in turns, as is natural. The first six eigenvalues $\lambda_n(a, 1-a)$ $(n = 0, 1, \ldots, 5)$ for $a \in [0, 1/2]$ are presented in Fig.2; as was noted $\lambda_n(a, 1-a) = \lambda_n(1-a, a)$, $\lambda_n(0, 1) = \lambda_n(1, 0) = \lambda_n^{f}$.

After the eigenvalues (numbers) $\lambda_n(a, b)$ have been determined, the eigenfunctions $S_n^{a,b}(x)$ can be represented according to (2.2) and (2.3) in the form

$$S_0(x) = x, \quad x \in [-a, b] \tag{2.7}$$

$$S_n^{a}(x) = -s(\alpha_n)\, q(\alpha_n + \lambda_n x)/q(\alpha_n) + s(\alpha_n + \lambda_n x), \quad x \in [-a, 0]$$

$$S_n^{b}(x) = [s(\beta_n)\, q(\beta_n - \lambda_n x)/q(\beta_n) - s(\beta_n - \lambda_n x)]\,[q(\alpha_n) - s(\alpha_n)\, r(\alpha_n)/q(\alpha_n)]\,[q(\beta_n) - s(\beta_n)\, r(\beta_n)/q(\beta_n)]^{-1}, \quad x \in [0, b]$$

$$\alpha_n = \lambda_n a, \quad \beta_n = \lambda_n b, \quad S_{-n}^{a,b}(x) = -S_n^{a,b}(x), \quad n = 1, 2, \ldots$$

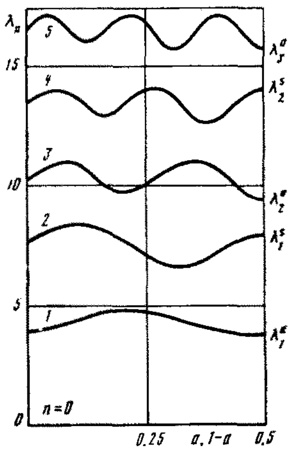

Fig.2

To fix our ideas, we set $D_a = 1$ in (2.7) and the coefficient D_b is taken from the first relationship in (2.3). Other equivalent representations are possible. The function $S_0(x)$ is the zeroth mode and corresponds to rod rotation without elastic displacements (as absolutely stiff: $v(t, x) \equiv 0$).

It follows from (2.7) that the eigenfunctions $S_n^{a,b}(x)$ satisfy the boundary conditions (2.1) by construction, and moreover, $S_n^{a}(x) \equiv 0$ for $a = 0$ and $S_n^{b}(x) \equiv 0$ for $b = 0$. The functions $S_n(x)$ $(x \in [-a, b]$, $n = 1, 2, \ldots)$ are analytic in the segments $x \in [-a, 0)$ and $x \in (0, b]$ and have continuous derivatives to second order inclusive at the point $x = 0$. We draw attention to the fact that it is sufficient to take the functions $S_n(x)$ $(n = 0, 1, 2, \ldots)$.

The system of eigenfunctions $\{S_n(x)\}$ (2.7) for the self-adjoint boundary value problem (2.1) is orthogonal in the ordinary sense /4, 6/

$$(S_n, S_m) = \int_{-a}^{b} S_n(x) S_m(x)\, dx = \int_{-a}^{0} S_n^{a}(x) S_m^{a}(x)\, dx + \int_{0}^{b} S_n^{b}(x) S_m^{b}(x)\, dx = \tag{2.8}$$

$$\| S_n \|^2 \delta_{nm}, \quad (n, m = 0, 1, 2, \ldots), \quad \| S_n \|^2 = \int_{-a}^{0} S_n^{a2}(x)\, dx + \int_{0}^{b} S_n^{b2}(x)\, dx$$

$$\sigma_n(x) = S_n(x) / \| S_n \|, \quad (\sigma_n, \sigma_m) = \delta_{nm}, \quad n, m \geqslant 0$$

Indeed, we multiply relationship (2.1) for $S = S_n^{a,b}$, $\lambda = \lambda_n$ by $S_m^{a,b}$, we integrate with respect to x in the appropriate intervals of x variation and add. Integrating by parts and using the boundary conditions (2.1) for $S_{n,m}^{a,b}$, we obtain (2.8).

The norms $\| S_n \|$ of the eigenfunctions $S_n(x)$ utilized in (2.8) to construct the orthonormalized system $\{\sigma_n(x)\}$, depend symmetrically on the parameters a and $b = 1 - a$ relative to $a = 1/2$ and commutation of the arguments like $\lambda_n(a, b)$. Consequently, it is sufficient to construct their graphs in the interval $a \in [0, 1/2]$. An analytic represenation of the function $\| S_n \|$ from $a, b = 1 - a$ is extremely awkward for $n \geqslant 1$, in particular

$$\| S_0 \|^2 = 1/3\,(a^3 + b^3) = (a^2 - a + 1/3) \geqslant 1/12$$

Therefore, the desired systems of eigenvalues $\{\lambda_n\}$ and the orthonormalized functions $\{\sigma_n(x)\}$ for the boundary value problem (2.1) are constructed in conformity with (2.4), (2.7) and (2.8). For $a = 0, 1/2, 1$ the system $\{\sigma_n(x)\}$ is a known complete orthonormalized system, i.e., the bases in the space $L^2[-a, b]$. The completeness in the general case is proved on the basis of the theory of integral operators /6/. The properties of uniform convergence and differentiability of the Fourier series corresponding to narrower classes of functions are established in the same way as the Steklov theorem /7/ and are later used to solve the initial problem (1.1), (1.3), (1.4) and (1.6).

3. *Solution of the problem of the motion of an elastic rod for given forces and moments.*

The desired function $u(t, x)$ is constructed by the Fourier method /2, 4, 5, 7/ on the basis of a complete orthonormalized system (basis) $\{\sigma_n(x)\}$. Grinberg's method /5/ used below is the following. According to (1.1) (for $l = \rho = EI = 1$) and (2.8), we obtain a denumerable system of equations for the unknown Fourier coefficients $\Theta_n(t)$ $(n = 0, 1, \ldots)$ of the function $u(t, x)$ in the basis $\{\sigma_n(x)\}$

$$\Theta_n^{\cdot\cdot} = -\int_{-a}^{0} u^{IV}\sigma_n^{a}(x)\,dx - \int_{0}^{b} u^{IV}\sigma_n^{b}(x)\,dx, \quad n = 0, 1, 2, \ldots \tag{3.1}$$

Integrating the expressions on the right in (3.1) by parts and using the boundary conditions (1.3), (1.4), and (2.1) for $u_{a,b}(t, x)$ and $\sigma_n^{a,b}(x)$, we obtain a denumerable system of equations and initial values for the unknown variables $\Theta_n(t)$:

$$\Theta_n^{\cdot\cdot} + \lambda_n^4\Theta_n = F_n(t), \quad t \in [0, T], \quad n = 0, 1, 2, \ldots \tag{3.2}$$

$$F_n(t) \equiv \sigma_n^{b}(b)\,P_B(t) - \sigma_n^{a}(-a)\,P_A(t) +$$
$$\sigma_n'(0)\,M_0(t) + \sigma_n^{a'}(-a)\,M_A(t) - \sigma_n^{b'}(b)\,M_B(t)$$
$$\Theta_n(0) = \Theta_n^0 \equiv f_n = (f, \sigma_n), \quad \Theta_n^{\cdot}(0) = \Omega_n^0 \equiv g_n = (g, \sigma_n)$$

Here f_n, g_n are Fourier coefficients of the functions $f(x)$, $g(x)$ in (1.6) in the basis $\{\sigma_n(x)\}$. If the functions $P_{A,B}(t)$, $M_{O,A,B}(t)$ are given, as is indeed assumed, then we determine the desired $\Theta_n(t)$, $\Theta_n^{\cdot}(t)$ in an elementary way:

$$\Theta_n(t) = f_n\cos\omega_n t + \frac{g_n}{\omega_n}\sin\omega_n t + \frac{1}{\omega_n}\int_0^t \sin\omega_n(t-\tau)\,F_n(\tau)\,d\tau \tag{3.3}$$

$$\Theta_n^{\cdot}(t) = d\Theta_n(t)/dt; \qquad \omega_n = \lambda_n^2(a, b), \qquad n = 0, 1, 2, \ldots$$

The expressions $\Theta_0(t)$, $\Theta_0^{\cdot}(t)$ are obtained from (3.3) for $n = 0$ by passing to the limit $\omega_n \to 0$. On the basis of the coefficients $\Theta_n(t)$ (3.3) obtained, the desired solution $u(t, x)$ of the problem formulated in Sect.1 is constructed as

$$u(t, x) = \sum_{n=0}^{\infty}\Theta_n(t)\,\sigma_n(x), \quad t \in [0, T], \quad x \in [-a, b] \tag{3.4}$$

$$\varphi(t) = u'(t, 0) = \sum_{n=0}^{\infty}\sigma_n'(0)\,\Theta_n(t)$$

$$v(t, x) = u(t, x) - \varphi(t)\,x = \sum_{n=1}^{\infty}\Theta_n(t)\,[\sigma_n(x) - x\sigma_n'(0)]$$

The dependence of the solution $u(t, x)$ (3.4) on the parameters of the problem a, $b = 1 - a$ is not indicated for brevity. We note that if the initial deviations and rates are vanishingly small while the dimensional frequencies are asymptotically large, the relationships (3.4) describe the rotation of an absolutely rigid rod

$$u(t, x) = \varphi(t)\,x, \quad v(t, x) \equiv 0, \quad t \in [0, T], \; x \in [-a, b] \tag{3.5}$$

$$\varphi(t) = \varphi^{\circ} + \omega^{\circ}t + J^{-1/2}(a, b)\int_0^t (t-\tau)\,F_0(\tau)\,d\tau$$

$$J(a, b) = \| S_0 \|^2 = {}^1/_3\,(a^3 + b^3) = a^2 - a + {}^1/_3, \quad {}^1/_{12} \leqslant J \leqslant {}^1/_3$$

$$F_0(t) = J^{-1/2}(a, b)\,M_\Sigma(t), \quad M_\Sigma \equiv bP_B + aP_A + M_O + M_A - M_B$$

The coefficient $J(a, b)$ in (3.5) has the meaning of a dimensionless moment of inertia of an absolutely rigid thin rod of unit length and density relative to the point $x = a$, $a \in [0, 1]$.

All the modes of the partial elastic rod vibrations turn out, according to (3.2), to be coupled general external actions $P_{A,B}(t)$, $M_{O,A,B}(t)$ that can be considered as controls and are selected from the requisite properties of the motion. According to /8/, the denumerable system (3.2) is controlled in a finite time interval $0 \leqslant t \leqslant T < \infty$. However, in contrast to a distributed control, the structural construction of a control by means of a finite number of control functions $P_{A,B}(t)$, $M_{O,A,B}(t)$ causes difficulties in principle /2, 3, 9, 10/. The so-called finite-mode approximation /3/ for which the coefficients $\Theta_n(t)$ $(n = 0, 1, 2, \ldots, n_{max})$ are taken into account is used as the basic practical approach. Appropriate controls are substituted into the initial system and their influence on the higher vibrations modes is estimated $(n > n_{max})$.

The influence of perturbations or the technical realization of the controlling effects $P_{A,B}(t)$, $M_{O,A,B}(t)$ can result in a direct connection between the variables. Θ_n $(n = 0, 1, 2, \ldots)$.

For instance, the controlling moment of the forces M_O relative to OZ often has the form $M_O = m(\varphi, \varphi^{\cdot}) + \mu(\varphi^{\cdot}, e(t))$, in applications, where m is the moment of the resistance force, μ is the electromagnetic moment, and e is the electrical voltage considered as the control of an electromechanical driver. Furthermore, the force effects $P_{A,B}$ can be realized by linear step motors in the form

$$P_{A,B} = P_{A,B}(u(t,x), u^{\cdot}(t,x), e_{A,B}(t))|_{x=-a,b} \approx P_{A,B}(x\varphi(t), x\varphi^{\cdot}(t), e_{A,B}(t))|_{x=-a,b}$$

Practical requirements result in the need to develop approximate methods for solving problems of controlling motions and their optimization for systems with distributed parameters, i.e., possessing significant elastic compliance of the elements and structures.

4. Estimates of elastic displacements and accuracy of positioning. For practical purposes it is usually interesting to formulate a problem of setting an elastic rod in a required angular position $(\varphi(T) = \varphi^T, \varphi^{\cdot}(T) = 0)$ or state of uniform rotation $(\varphi^{\cdot}(T) = \omega^T)$ without relative vibrations $(v(T, x) = v^{\cdot}(T, x) \equiv 0)$. Since the exact solution is not constructed successfully, the rod will have residual elastic displacments. Evaluation of the quantity $v(t, x)$ governing these deviations, enables different accuracy characteristics for control of the rod rotation to be estimated. To investigate such a problem it is convenient to change to the variables $v(t, x)$, $\varphi(t)$. We consequently obtain a system of partial integrodifferential equations

$$\begin{gathered} v^{\cdot\cdot} = -v^{\mathrm{IV}} - x\varphi^{\cdot\cdot}, \quad v = v(t,x) = v_{a,b}(t,x) \\ v_a(t,0) = v_a'(t,0) = v_b(t,0) = v_b'(t,0) = 0 \\ (-[v_b''(t,0) - v_a''(t,0)] = M_O(t)), \ t \in [0, T] \\ -v_a''(t,-a) = M_A(t), \qquad -v_a'''(t,-a) = P_A(t) \\ -v_b''(t,b) = M_B(t), \qquad -v_b'''(t,b) = P_B(t) \\ v(0,x) = h(x) \equiv f(x) - \varphi^0 x, \qquad v^{\cdot}(0,x) = k(x) \equiv g(x) - \omega^0 x \\ x \in [-a, b], \quad h(0) = h'(0) = k(0) = k'(0) = 0 \end{gathered} \tag{4.1}$$

$$\begin{gathered} J(a,b)\varphi^{\cdot\cdot} + \int_{-a}^{b} v^{\cdot\cdot}(t,x)\,x\,dx = M_\Sigma(t) \\ \varphi(0) = \varphi^0, \quad \varphi^{\cdot}(0) = \omega^0 \end{gathered} \tag{4.2}$$

Problem (4.1), (4.2) is equivalent to the original problem (1.1), (1.3), (1.4), (1.6). However, the kind of boundary value problems for v is different; the boundary conditions have changed. It can be solved by a method analogous to that mentioned in Sects.2 and 3.

We note that any eight of the nine boundary conditions can be selected from the system written. Moreover, (4.2), which contains the integral of $v^{\cdot\cdot}(t,x)\,x$, is equivalent to the following by virtue of (4.1):

$$\begin{gathered} -v_b'''(t,b) - v_a'''(t,-a) - [v_b''(t,0) - v_a''(t,0)] - \\ v_a''(t,-a) + v_b''(t,b) = M_\Sigma(t), \quad t \in [0, T] \end{gathered} \tag{4.3}$$

Substitution of the boundary values according to (4.1) into (4.3) results in an identity, as might have been expected. If the function $\varphi^{\cdot\cdot} = \gamma(t)$ (a "kinematic" control) is given, then the relationship (4.3) is a condition on the force actions, which when satisfied results in realization of a given rotational motion of the tangent Ox (the "direct" dynamics problem). We note that the left side of (4.3) is a linear integral operator of $\gamma, P_{A,B}, M_{O,A,B}$. Furthermore, if the functions $P_{A,B}(t)$, $M_{O,A,B}(t)$ are given, relation (4.3) is a linear Volterra-type integral equation with difference kernel of the first kind /11/ with all the singularities of matching the smoothness and the order of zeros. It follows from the structure of the solution obtained by the Fourier method that the integral equation has the form

$$\int_0^t L(t-\tau)\gamma(\tau)\,d\tau = H(t), \quad L(t) \equiv \sum_{n=1}^{\infty} \frac{L_n}{\Omega_n} \sin \Omega_n t \tag{4.4}$$

Here L_n are Fourier coefficients of the function $l(x) \equiv x$ in a basis generated by the boundary value problem (4.1); Λ_n, $\Omega_n = \Lambda_n^2$ are the eigenvalues and frequencies, respectively. The function $H(t)$ is defined in terms of the known $P_{A,B}(t)$, $M_{O,A,B}(t)$ and the Fourier coefficients of the functions $h(x), k(x)$. We note that the kernel of the integral operator vanishes for $t = \tau$, which requires matching of the order of the zeros of the right side of

$H(t)$ if the solution $\gamma(t)$ is constructed in the class of continuous (not generalized) functions /11/, as is dictated by the physical conditions of strength.

A typical formulation of the ("dynamic") control problem for system (4.1) and (4.2) is the selection of allowable controls (and optimal controls in a certain quality criterion) $P_{A,B}(t)$ and $M_{O,A,B}(t)$ setting the system for $t = T$ in the required state of rotation or rest as a whole (without elastic displacements $v(T, x) = v^{\cdot}(T, x) \equiv 0$). This state will be conserved for $t > T$ if the controls are set equal to zero.

The state of the static deflection $v = v^0(x)$ of a rod for which $v^{\cdot}(t, x) \equiv 0$, $\gamma(t) = \gamma^0 =$ const is of interest in practice. It is realized for constant controlling actions $P_{A,B}$ and $M_{O,A,B}$, in particular $\gamma(t) \equiv 0$, if $M_\Sigma(t) \equiv 0$ here. Integrating the static deflection equation we obtain

$$v_{a,b}^0(x) = \frac{x^2}{2!}\left(-\frac{1}{3}l_{a,b}^3\gamma^0 - M_{A,B} + l_{a,b}P_{A,B}\right) + \frac{x^3}{3!}\left(\frac{1}{2}l_{a,b}^2\gamma^0 - P_{A,B}\right) - \frac{x^5}{5!}\gamma^0, \quad \gamma^0 = \frac{M_\Sigma}{J(a,b)} \tag{4.5}$$

$$l_a = -a, \quad l_b = b; \quad \varphi^{\cdot\cdot} = \gamma^0, \quad M_\Sigma, P_{A,B} = M_{O,A,B} = \text{const}$$

Conditions on the control for which $v_a^0(x) \equiv 0$ $(M_A = P_A = \gamma \equiv 0)$, but $v_b^0(x) \not\equiv 0$ follow simply from (4.5); analogously $v_b^0(x) \equiv 0$ (if $M_B = P_B = \gamma \equiv 0$), but $v_a^0(x) \not\equiv 0$. The whole rod remains straight if and only if all the functions $P_{A,B}(t) = M_{O,A,B}(t) \equiv 0$, as is obvious. The elementary expressions (4.5) are useful when $T \gg T_1$, where $T_1 = 2\pi/\Omega_1$ is the period of the lowest elastic vibrations mode. In the case of smooth, practically constant controls, the natural vibrations damp out and rod rotation will be described by the relationship (4.5).

The orthonormalized eigenfunctions $\sigma_n(x)$ (3.2), (3.3) and the Fourier coefficients $\Theta_n(t)$ constructed according to (2.7) and (2.8) and the eigenvalues $\lambda_n(a, b)$ and frequencies $\omega_n = \lambda_n^2$ calculated from (2.4) (see (2.5), (2.6), and (3.3) also) permit the simple construction of rational control laws, the estimation of the accuracy of the motion being obtained in each specific case with respect to the required motion and the constructive selection of the composition of the controlling effects and the parameter a to improve the quality of the control process.

REFERENCES

1. RABOTNOV YU.N., Mechanics of a Deformable Solid. Nauka, Moscow, 1979.
2. AKULENKO L.D. and GUKASYAN A.A., Control of plane motions of an elastic manipulator linkage, Izv. Akad. Nauk SSSR, Mekhan. Tverd. Tela, 5, 1983.
3. AKULENKO L.D. and BOLOTNIK N.N., The control of the rotation of the circular section of a manipulator, Izv. Akad. Nauk SSSR, Tekh. Kibernetika, 1, 1984.
4. TIKHONOV A.N. and SAMARSKII A.A., The Equations of Mathematical Physics. Nauka, Moscow, 1966.
5. GRINBERG G.A., A new method of solving certain boundary value problems for equations of mathematical physics allowing separation of variables, Izv. Akad. Nauk SSSR, Ser. Fiz., 10, 2, 1946.
6. RIESZ F. and SZEKEFALVY-NAGY B., Lectures on Functional Analysis, Mir, Moscow, 1979.
7. KRYLOV A.N., Collected Work, 3, 2, Izd. Akad. Nauk SSSR, Moscow, 1949.
8. POLTAVSKII L.N., On the finite controllability of infinite systems of pendulums. Dokl. Akad. Nauk SSSR, 245, 6, 1979.
9. BUTKOVSKII A.G., Methods of Controlling Systems with Distributed Parameters. Nauka, Moscow, 1975.
10. AKULENKO L.D., Reduction of an elastic system in a given state by a force boundary action, PMM, 45, 6, 1981.
11. KRASNOV M.L., KISELEV A.I. and MAKARENKO G.I., Integral Equations, Nauka, Moscow, 1968.

Translated by M.D.F.